The Five Lemma for Abelian Categories and the Phenomenology of Mathematical Abstraction

Micah Phillips-Gary

The College of Wooster

The Five Lemma for Groups

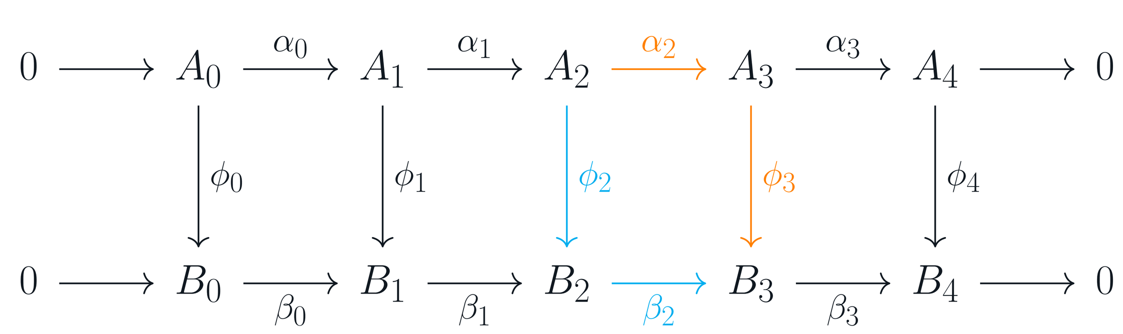
A **group** is a set G with a binary operation such that there is an identity element and inverses.

Example 1 *The integers are a group. For any integer z, $z + 0 = z$. Also, there exists some $-z$ such that $-z + z = 0$.*

The **kernel** of a function between groups is the set of elements that get mapped to the identity.

A sequence of functions between groups is said to be **exact** when the composition of one function with the next takes every element to the identity.

Theorem 1 (Five Lemma for Groups) *Given the following commutative diagram with top and bottom rows exact sequence, if ϕ_0, ϕ_1, ϕ_3 and ϕ_4 are **isomorphisms** (bijective homomorphisms), then so is ϕ_2.*

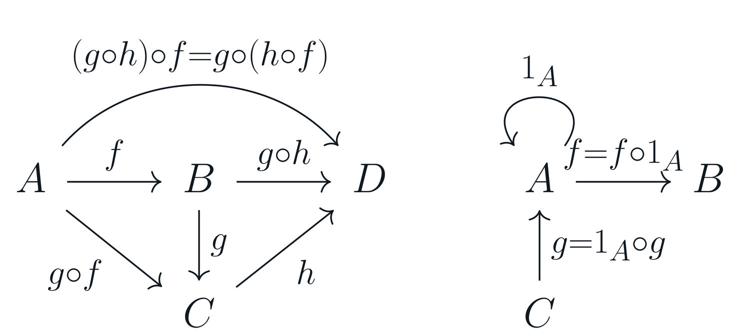


In group theory, we prove this theorem via **diagram chasing**. That is, we take elements in A_2 (to show injectivity) or B_2 (to show surjectivity) and "chase" them around the diagram by looking for the elements that get mapped to them and which they, in turn, get mapped to. We do this via commutativity relations like the one given by the highlighted arrows.

Category Theory

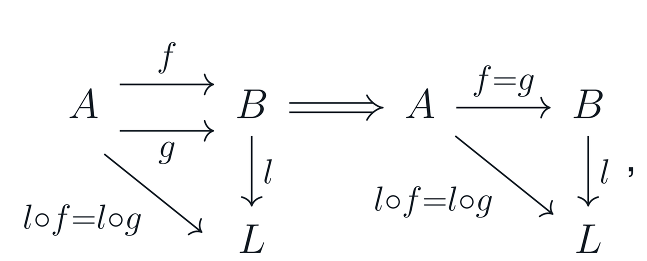
"The purview of category theory is *mathematical analogy*. [...] The category-theoretic perspective can function as a simplifying abstraction, isolating propositions that hold for formal reasons from those whose proofs require techniques particular to a given mathematical discipline." [Riehl, 2016, p. ix].

A **category** $\mathcal{C}$ is a collection of objects with **morphisms** between them. There must be an operation defined on these morphisms such that the following properties hold:

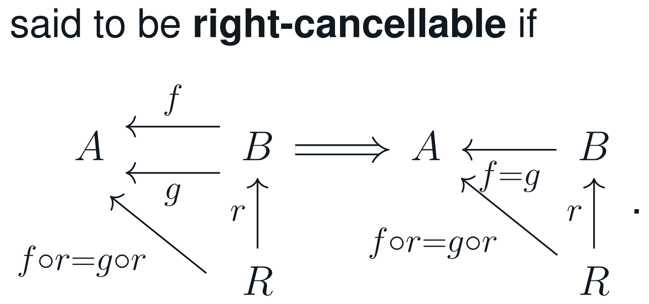


We write $f \in hom(A, B)$ as $f : A \to B$. Two morphisms f and g are said to **parallel** if $f : A \to B$ and $g : A \to B$.

A morphism l is said to be **left-cancellable** if



while a morphism r is said to be **right-cancellable** if



Notice that the diagrams which define "left-cancellable" and "right-cancellable" are the same, simply with the directions of morphisms reversed. This makes them **dual** concepts.

Finally, if $f : A \to B$, then f is said to be an **isomorphism** if for some $f^{-1} : B \to A$, $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

Abelian Categories

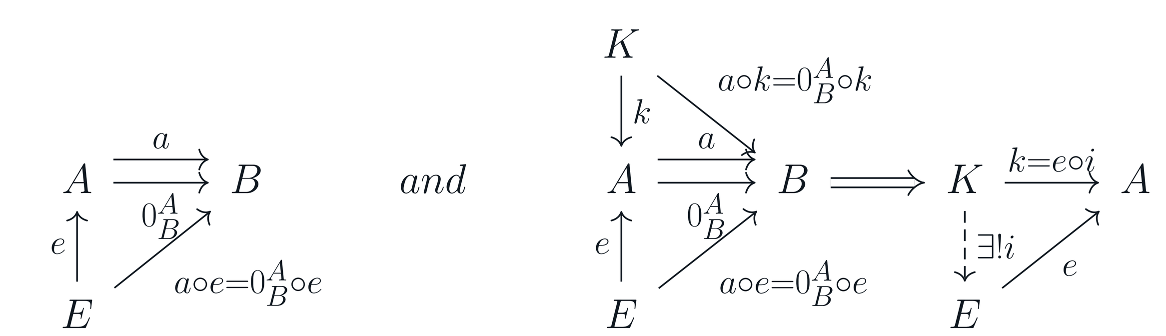
A category $\mathcal{A}$ is said to be **abelian** if:

1. $\mathcal{A}$ is preadditive; i.e., all hom-sets are groups such that $a+b = b+a$ for any two parallel morphism a, b.
2. $\mathcal{A}$ contains a **zero object** (an object 0 such that there is exactly one morphism from any object A to 0 and exactly one morphism to any object A from 0).
3. $\mathcal{A}$ contains biproducts for any two objects.
4. $\mathcal{A}$ contains kernels (and the dual concept, cokernels) for any morphism.
5. All left-cancellable morphisms in $\mathcal{A}$ are kernels and all right-cancellable morphisms are cokernels.

We can prove the Five Lemma for Abelian Categories but not for generic categories.

Kernel, Cokernel, Image

Let $a : A \to B$. By definition of zero object there exist some morphisms $z_1 : A \to 0$ and $z_2 : 0 \to B$, such that $z_2 \circ z_1 = 0^A_B$ is parallel with a. We define the **kernel** of a as a morphism e such that



The dual concept of a kernel is that of a **cokernel**. To define it, simply reverse the direction of the morphisms in the above diagram.

The basic category theoretic definition of the **image** is simply the kernel of the cokernel of a morphism. An **exact sequence** is then a sequence of morphisms $\alpha_0 : A_0 \to A_1$, $\alpha_1 : A_1 \to A_2$, ... where the kernel of α_{n+1} and the image of α_n are **equivalent as subobjects** of A_{i+1}. This means that for some morphisms m, n if $k(\alpha_{i+1})$ is the kernel of α_{i+1} and $kc(\alpha_i)$ is the image of α_i, $k(\alpha_{i+1}) \circ n = kc(\alpha_i)$ and $kc(\alpha_i) \circ m = k(\alpha_{i+1})$.

Rules for Diagram Chasing

To prove the Five Lemma requires the redefinition of these concepts in terms of member equivalence. Two morphisms $x \to A$ and $u \to A$ are said to be **member equivalent** (denoted $x \equiv y$) if $x \circ u = y \circ v$ for some right-cancellable morphisms u and v.

Theorem 2 *The following statements are equivalent in an abelian category:*

- *$f : A \to B$ and $g : B \to C$ form an exact sequence.*
- *$kc(f) \equiv k(g)$.*
- *$g \circ f = 0^A_C$ and, for all $y \to B$ such that $g \circ y \equiv 0^Y_C$, there exists some $x \to A$ such that $f \circ x \equiv y$.*

Theorem 3 *In an abelian category, $f : A \to B$ is right-cancellable if and only if for each $x \to B$ there exists some $y \to A$ such that $f \circ y \equiv x$.*

Theorem 4 *The following statements are equivalent in an abelian category:*

- *$f : A \to B$ is left-cancellable.*
- *for all $x \to A$, $f \circ x \equiv 0^P_B$ only if $x \equiv 0^Q_A$.*
- *for all $x, y \in_m A$, $f \circ x \equiv f \circ y$ only if $x \equiv y$.*

Because a morphism that is both left- and right-cancellable in an Abelian category will be an isomorphism, these three theorems provide the necessary equivalent definitions to prove the Five Lemma for Abelian Categories in an analogous way to that in which we prove the Five Lemma for Groups (i.e., by diagram chasing).

Phenomenological Conclusions

Phenomenology is the philosophical study of lived experience as such, including the lived experience of the mathematician. The father of phenomenology, Edmund Husserl, distinguished between the generalization and the formalization or abstraction of a proposition [Husserl, 2014, p. 27]. In **generalization**, any specific concept is replaced by the genus it falls under (e.g., "falcon" is replaced by "bird"). In **abstraction**, all material concepts are emptied out, no matter how general, leaving an empty logical form (e.g., "falcon" is replaced by "subject" or "predicate" depending on which role it plays in the sentence).

Here are some insights we can draw by applying Husserl's distinction to category theory:

1. Category theory's relation to lower-level areas of mathematics (such as group theory) can be best understood as a kind of abstraction.
2. Hence, the Five Lemma for Abelian Groups is both a materialization of the Five Lemma for Abelian Categories and a specification of the Five Lemma for Groups.
3. There are relations of both generalization/specification and abstraction/materialization between concepts within mathematics.
4. Hence, the distinction between abstract logical form and conceptual content is relative, not absolute, because mathematics already deals only with abstract logical form.

References

[Husserl, 2014] Husserl, E. (2014). *Ideas I.* Hackett, Indianapolis, IN. Dahlstrom, D., translator.

[Riehl, 2016] Riehl, E. (2016). *Category Theory in Context.* Dover Publications, Mineola, New York.